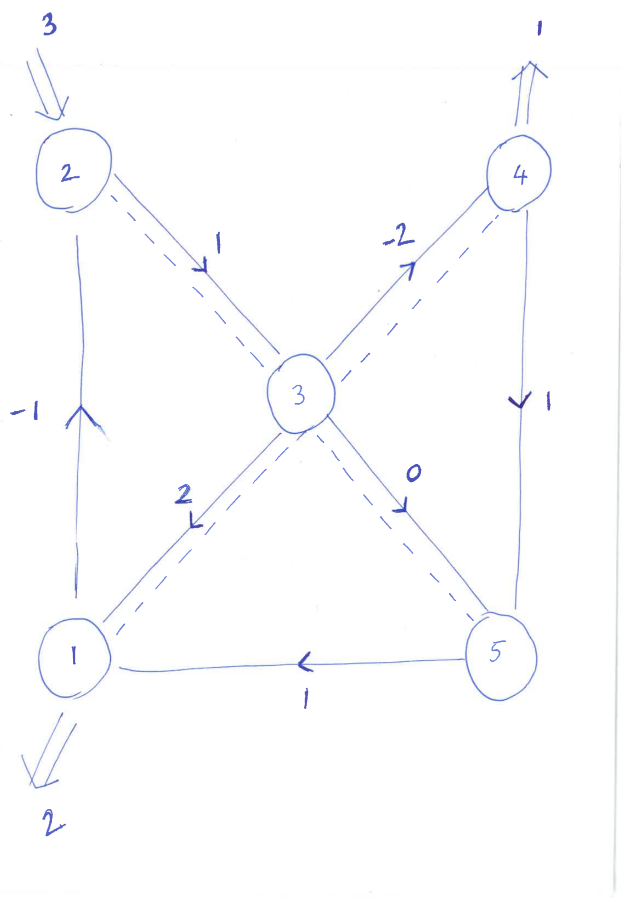
Final - Optimization (2019-20)
Time: 3 hours.

Attempt all questions. There are a total of 55 points, the maximum you can score is 50.

1. Consider the *uncapacitated* network flow problem on a directed graph with supplies. Show that if a flow vector is *not* a tree solution, then it is *not* a basic solution of the problem. **[5 marks]**

2. Consider the network flow problem on the directed graph shown below. The numbers next to each directed arc $\to$ is the *cost* associated to the arc, while the numbers next to $\Rightarrow$ is the external *supply/demand* at the node.



Denote by $\mathbf{c}$ the vector of costs corresponding to the arc. We are interested in minimizing the total cost $\mathbf{c}^T\mathbf{f}$, where the flow vector $\mathbf{f}$ satisfies the flow conservation equations and $\mathbf{f} \geq \mathbf{0}$.

   (a) Find an optimal basic feasible solution (feasible tree solution) to the problem. **[7 marks]**

   (b) Find the optimal cost for the problem. **[3 marks]**

Note: If you are implementing the simplex algorithm, you can find an initial feasible basic/tree solution by considering the tree indicated by dashed lines.

3. Let $\mathbf{A}$ be a symmetric square matrix. Consider the linear programming problem: minimize $\mathbf{c}^T\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} \geq \mathbf{c}$, $\mathbf{x} \geq \mathbf{0}$. Prove that if $\mathbf{x}^*$ satisfies $\mathbf{A}\mathbf{x}^* = \mathbf{c}$ and $\mathbf{x}^* \geq \mathbf{0}$, then $\mathbf{x}^*$ is an optimal solution. **[5 marks]**

4. Give an example of a linear program of the form $\min_{\mathbf{x}\in P} \mathbf{c}^T\mathbf{x}$ where the polyhedron P has an extreme point but the minimization problem is unbounded. **[3 marks]**

5. Consider the primal problem: minimize $x_1 + 4x_2$

$$\begin{aligned} \text{subject to} \qquad 2x_1 + x_2 &\geq 6, \\ 5x_1 + 3x_2 &\geq 7, \\ x_1, x_2 &\geq 0. \end{aligned}$$

(a) Write down the dual of the above problem. **[3 marks]**

(b) Find optimal solutions to the primal and the dual problems. **[5 marks]**

(c) Find the optimal costs of the primal problem and the dual problems. **[2 marks]**

6. Solve the linear program: minimize $-2x_1 - x_2 + x_3 - 4x_4$

$$\begin{aligned} \text{subject to} \qquad x_1 - x_2 + x_3 - 3x_4 &= 2, \\ x_1 + x_2 + 3x_4 &= 6, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

[5 marks]

7. Let $\mathbf{A}$ be an $n \times n$ real matrix. Then show that the following are equivalent: **[5 marks]**

- $\mathbf{A}$ is orthogonal, that is $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.
- $\|\mathbf{Ax}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbf{R}^n$, where $\|\cdot\|$ is the Euclidean norm in $\mathbf{R}^n$.

8. While solving a linear programming problem (in standard form) by the simplex method, the following tableau is obtained at some iteration.

	0	$\cdots$	0	$\bar{c}_{m+1}$	$\cdots$	$\bar{c}_n$
x_1	1	$\cdots$	0	$a_{1,m+1}$	$\cdots$	$a_{1,n}$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
x_m	0	$\cdots$	1	$a_{m,m+1}$	$\cdots$	$a_{m,n}$

Assume that in this tableau we have $\bar{c}_j \geq 0$ for $j = m+1, \cdots, n-1$ and $\bar{c}_n < 0$. In particular, x_n is the only candidate for entering the basis.

(a) Suppose that x_n indeed enters the basis and that this is a nondegenerate pivot (that is, $\theta^* \neq 0$). Prove that x_n will remain basic in all subsequent iterations of the algorithm and that x_n is a basic variable in any optimal basis. **[7 marks]**

(b) Suppose that x_n indeed enters the basis and that this is a degenerate pivot (that is, $\theta^* = 0$). Show that x_n need not be basic in an optimal basic feasible solution. **[5 marks]**